Math 259A Lecture 7 Notes

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1 WO and SO Continuity of Linear Functionals and The Pre-Dual of $\mathcal{B}$

1.1 Weak operator and strong operator continuity of linear functionals

Lemma 1.1. *Let X be a vector space with seminorms $p_1, \dots, p_n$. Let $\varphi : X \to \mathbb{C}$ be a linear functional such that $|\varphi(x)| \leq \sum_{i=1}^n p_i(x)$ for all $x \in X$. Then there exist linear functionals $\varphi_1, \dots, \varphi_n : X \to \mathbb{C}$ such that $\varphi = \sum_i \varphi_i$ with $|\varphi_i(x)| \leq p_i(x)$ for all $x \in X$ and for all i.*

Proof. Let $D = \{\tilde{x} = (x, \dots, x) : x \in X\} \subseteq X^n$, which is a vector subspace. On X^n, take $p((x_i)_{i=1}^n) = \sum_i p_i(x_i)$. We also have a linear map $\tilde{\varphi} : D \to \mathbb{C}$ given by $\tilde{\varphi}(\tilde{x}) = \varphi(x)$. This map satisfies $|\tilde{(}\tilde{x})| \leq p(\tilde{x})$. By the Hahn-Banach theorem, there exists an extension $\psi \in (X^n)^*$ of $\tilde{\varphi}$ such that $|\psi(x_1, \dots, x_n)| \leq p(x_1, \dots, x_n)$. Now define $\varphi_k(x) := \psi(0, \dots, x, 0, \dots)$, where the x is in the k-th position. $\square$

Theorem 1.1. *Let $\varphi : \mathcal{B} \to \mathbb{C}$ be linear. φ is weak operator continuous if and only if it is it is strong operator continuous.*

Proof. We only need to show that if φ is strong operator continuous, then it is weak operator continuous. So assume there exist $\xi_1, \dots, \xi_n \in X$ such that $|\varphi(x)| \leq \sum_{i=1}^n \|x\xi_i\|$ for all $x \in \mathcal{B}$. By the lemma, we can split $\varphi = \sum \varphi_k$, such that $|\varphi_k(x)| \leq \|x\xi_k\|$ for all x and k. By the Riesz representation theorem, there exists an $\eta_k \in H$ such taht $\varphi_k(x) = \langle x\xi_k, \eta_k \rangle$. So $\varphi(x) = \sum_k \langle x\xi_k, \eta_k \rangle$. So φ is weak operator continuous. $\square$

Corollary 1.1. *Any closure in $\mathcal{B}(H)$ of a convex set is the same with respect to the weak operator and strong operator topologies.*

Proof. If we have a point in the closure wrt one topology but not in the other, we can separate it with a hyperplane using the geometric Hahn-Banach theorem. $\square$

Corollary 1.2. *Let $M \subseteq \mathcal{B}(H)$ be a vector subspace. Then a linear functional $\varphi : M \to \mathbb{C}$ is weak operator continuous if and only if it is strong operator continuous.*

1.2 The pre-dual of $\mathcal{B}$

Recall that we had $\mathcal{B}_\sim = \text{span}\{x \mapsto \sum_k \langle x\xi_k, \eta_k \rangle : \xi_k \eta_k \in H\}$ and $\mathcal{B}_* := \overline{\mathcal{B}_\sim}$. This is the same as taking finite rank operators in $T \in \mathcal{B}(H)$ and taking the functionals $x \mapsto \text{tr}(xT)$.

Remark 1.1. $\mathcal{B}_*$ is the space of trace class operators. Suppose $T \in \mathcal{B}(H)$ has finite rank. We have $|T| = (T^*T)^{1/2}$ by functional calculus. Then $\text{tr}(|T|)$ is the **Schatten-von Neumann 1-norm** of the operator.

Theorem 1.2. $\mathcal{B} = (\mathcal{B}_*)^*$ *via the duality pairing* $\mathcal{B} \times \mathcal{B}_* \to \mathbb{C}$ *given by* $\langle x, \varphi \rangle = \varphi(x)$.

Here is the idea: Since $\mathcal{B}_* \subseteq \mathcal{B}^*$, we can view its elements as linear functionals on $\mathcal{B}$. But then we can view elements of $\mathcal{B}$ as linear functionals on $\mathcal{B}_*$.

Proof. If $x \in \mathcal{B}$, denote $\Phi_x : \mathcal{B}_* \to \mathbb{C}$ by $\Phi_x(\varphi) = \varphi(x)$. Then $|\Phi_x(\varphi)| \leq \|\varphi\| \cdot \|x\|$, soo $\|\Phi_x\| \leq \|x\|$. So the map $\mathcal{B} \to (\mathcal{B}_*)^*$ sending $x \mapsto \Phi_x$ is a contraction. In fact, $\|\Phi_x\| = \|x\|$ because $\|x\| = \sup_{\xi,\eta \in (H)_1} |\langle x\xi, \eta \rangle$.

To show that this is surjective, take $\Phi \in (\mathcal{B}_*)^*$. Then consider the map $H \times H \mapsto \mathbb{C}$ given by $(\xi, \eta) \mapsto \Phi(\omega_{\xi,\eta})$, where $\omega_{\xi,\eta} = \langle \cdot \xi, \eta \rangle$. So by Riesz-representation, there is an $x \in \mathcal{B}$ such that $\Phi(\omega_{\xi,\eta}) = \langle x, \xi, \eta \rangle$. So $\Phi = \Phi_x$. $\square$

Corollary 1.3. $(\mathcal{B})_1$ *is weak operator compact.*

Proof. This is the topology $\sigma(\mathcal{B}, \mathcal{B}_*)$ topology on $(\mathcal{B})_1$. By the Banach-Alaoglu theorem, this is compact. $\square$

Corollary 1.4. *Let* $M \subseteq \mathcal{B}$ *be a vector subspace which is weak operator closed. Denote* $M_* = \{\varphi|_M : \varphi \in \mathcal{B}_*\}$. *Then* $(M_*)^* = M$ *via the duality pairing* $\langle x, \varphi \rangle = \varphi(x)$. *Thus, any von Neumann algebra is the dual of some space.*

Remark 1.2. Any C^*-algebra with a pre-dual is a von-Neumann algebra.

Here is some notation:

1. If $X \subseteq H$ is a nonempty subset, then we denote $[X]$ to be the norm closure of span X. We may also use this same notation to denote the orthogonal projection of that space (but this will be clear in context).

2. Let $S \subseteq B(H)$ be nonempty. Then we denote $S' = \{x \in \mathcal{B}(H) : xy = yx \, \forall y \in S\}$ to be the **commutant** of S in $\mathcal{B}(H)$.

Proposition 1.1. S' *is strong operator closed, and it is an algebra. If* $S = S*$, *where* $S^* = \{x^* : x \in S\}$, *then* S' *is a *-algebra. In this case,* S' *is weak operator closed and is hence a von-Neumann algebra.*

Proof. If $x_i \in S'$ and $x_i \xrightarrow{\text{so}} x \in B(H)$, then

$$xy\xi = \lim_i x_i y\xi = \lim_i y\xi_i\xi = y \lim_i x_i\xi = yx\xi,$$

so $x \in S'$. $\square$

Remark 1.3. Physicists view S' as an algebra of symmetries of states in a system.

We will prove the following theorem next time.

Theorem 1.3 (von Neumann's bicommutant theorem, 1929)**.** *Let $M \subseteq B(H)$ be a *-algebra with $1_M = \mathrm{id}_H$. Then M is a von Neumann algebra if and only if $M = (M')'$.*

Remark 1.4. Some people call this von Neumann's density theorem because it says that the weak operator closure of M is $(M')'$.

Theorem 1.4 (Kaplansky, late 50s)**.** *Let M, M_0 be *-algebras. If the strong operator closure of M_0^* equals M, then $(\overline{M_0}^{\text{so}})_1 = (M)_1$.*